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The problem

Introduction.

Let G be a finite set of elements and $\cdot$ a binary associative operation on G. There is a neutral element in G and that is the only element in G with the property $a \cdot a = a$.
Show that G with the operation $\cdot$ is a group.

Solution.

G is a finite semigroup with identity. Let A be a subset of G. There is a smallest subsemigroup K of G which contains A. We say A generates K, notation $\{A\} = K$. A single element x of G generates a subsemigroup $\{x\} = \{x^n | n > 0\}$. Since $\{x\}$ is finite there must be integers $p > q$, such that $x^p = x^q$. So $x^p = x^{q+k} = x^q x^k = x^k x^q = x^q$ and $e = x^k$ is a neutral element for $\{x\}$. We assume that k is the smallest integer with this property. We easily verify that $\{x\} = \{e, x, x^2, ..., x^{k-1}\}$ is a group with neutral element e and as such a subgroup of G. Clearly e is idempotent with $e \cdot e = e^2 = e$. According to the problem statement e is the only element of G with this property.

We now proof the following lemma:
Let G be a finitely generated semigroup and H een subgroup of G. Then there exists a maximal subgroup M of G containing H.

Proof: Let G be generated by $x_1, ..., x_m$ and let y_1 be the first of the x_i not contained in H and with property $H_1 = \{H, y_1\}$ is a group. If such a y_1 does not exist then $M = H$ is the maximal subgroup of G. We now have $H_1 \supseteq H$. If $H_1 = G$, then G is the maximal subgroup sought. If not, choose $H_2 = \{H_1, y_2\} \supseteq H_1$, where y_2 is the first of the x_i not contained in H_1 and

$\{H_1, y_2\}$ is a group. If such a y_2 does not exist then $M = H_1$ is the maximal subgroup of G.
Continuing this proces we must reach the situation where no more extension is possible: $H_i \supseteq H_{i-1} \supseteq ... \supseteq H$, H_i is a group. If $H_i = \{H_{i-1}, y_i\} = G$ the maximal subgroup is G else the maximal subgroup $M = H_i$ is a proper subgroup of G.

G is finite and so certainly finitely generated. According to the above lemma $\{x\}$ is contained in a maximal subgroup M. If $M = G$ we are ready, but let there be a y not in M, then $\{y\}$ is contained in a maximal subgroup M', with neutral element e', with $e' \cdot e' = e'$. If $e' \neq e$ we have a contradiction and there is no such element y, hence $M = G$. If $e' = e$ than we easily see that $\{M, y\}$ is a group in contradiction with the maximality of M. So we have proved that G is a group.